

Embedded loops in the hyperbolic plane with prescribed, almost constant curvature

Roberta Musina*

Fabio Zuddas[†]

Abstract

Given a constant $k > 1$ and a real valued function K on the hyperbolic plane \mathbb{H}^2 , we study the problem of finding, for any $\varepsilon \approx 0$, a closed and embedded curve u^ε in \mathbb{H}^2 having geodesic curvature $k + \varepsilon K(u^\varepsilon)$ at each point.

1 Introduction

Let Σ be an oriented Riemannian surface with empty boundary, Riemannian metric tensor g and Levi-Civita connection ∇^Σ . The geodesic curvature of a regular loop $u \in C^2(\mathbb{S}^1, \Sigma)$ is given by

$$K(u) = \frac{\langle \nabla_{u'}^\Sigma u', i_u u' \rangle_g}{|u'|_g^3}.$$

Here we denoted by $i_u : T_u \Sigma \rightarrow T_u \Sigma$ the isometry that rotates $T_u \Sigma$, in such a way that $\{\tau, i_u \tau\}$ is a positively oriented orthogonal basis of $T_u \Sigma$, for any $\tau \neq 0$.

Given a sufficiently smooth function $K : \Sigma \rightarrow \mathbb{R}$, the *K-loop problem* consists in finding regular curves $u \in C^2(\mathbb{S}^1, \Sigma)$ having geodesic curvature $K(u)$ at each point. This problem can be faced by studying the system of ordinary differential equations

$$\nabla_{u'}^\Sigma u' = L^\Sigma(u) K(u) i_u u', \quad L^\Sigma(u) := \left(\oint_{\mathbb{S}^1} |u'|_g^2 dx \right)^{\frac{1}{2}}. \quad (1.1)$$

*Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN project 2015KB9WPT_001.

[†]Dipartimento di Matematica e Informatica, Università di Cagliari, Palazzo delle Scienze, Via Ospedale 72 – 09124 Cagliari, Italy. Email: fabio.zuddas@unica.it.

Indeed, every nonconstant solution $u \in C^2(\mathbb{S}^1, \Sigma)$ to (1.1) has constant speed $|u'|_g = L^\Sigma(u)$, use for instance the computations in [14, Chapter 4]. Therefore u is regular, and has curvature $K(u)$ at each point.

The K -loop problem has been largely studied since the seminal work [4] by Arnol'd. Most of the available existence results require compact target surfaces Σ ; we limit ourselves to cite [9, 12, 13, 19, 20, 21, 22, 23] and references therein.

In the present paper we take Σ to be the (noncompact) hyperbolic plane \mathbb{H}^2 . It turns out that the problem under consideration does not have solutions, in general (see Subsection 2.2). In particular, if $-1 \leq K(q) \leq 1$ for any $q \in \Sigma$, then no K -loop exists. If $K \equiv k > 1$ is constant (recall that changing the orientation of a curve changes the sign of its curvature), then any regular parameterization of an hyperbolic circle of radius

$$\rho_k = \operatorname{artanh} \frac{1}{k} = \frac{1}{2} \ln \frac{k+1}{k-1}$$

is a k -loop; conversely, any k -loop in \mathbb{H}^2 parameterizes some circle of radius ρ_k .

Our existence results involve curvatures that are small perturbations of a given constant $k > 1$. In Section 3 we carefully choose a reference parameterization ω of a circle of radius ρ_k . Then we take any point $z \in \mathbb{H}^2$ and compose ω with an hyperbolic translation to obtain a parameterization ω_z of $\partial D_{\rho_k}^{\mathbb{H}}(z)$. Next, given $K \in C^1(\mathbb{H}^2)$, we look for a point $z_0 \in \mathbb{H}^2$ and for embedded $(k + \varepsilon K)$ -loops in \mathbb{H}^2 that suitably approach the circle ω_{z_0} as $\varepsilon \rightarrow 0$.

The center z_0 can not be arbitrarily prescribed. In fact, in Theorem 4.1 we prove that if there exists a sequence of $(k + \varepsilon_h K)$ -loops u_h such that $\varepsilon_h \rightarrow 0$ and $u_h \rightarrow \omega_{z_0}$ suitably, then z_0 is a critical point for the Melnikov-type function

$$F_k^K(z) = \int_{D_{\rho_k}^{\mathbb{H}}(z)} K(z) dV_{\mathbb{H}}, \quad F_k^K : \mathbb{H}^2 \rightarrow \mathbb{R}. \quad (1.2)$$

One may wonder whether the existence of a critical point z_0 for F_k^K is sufficient to have the existence, for $\varepsilon \approx 0$, of an embedded $(k + \varepsilon K)$ -loop $u_\varepsilon \approx \omega_{z_0}$. We can give a positive answer in case F_k^K has a *stable* critical point, accordingly with the next definition (see also [3, Chapter 2]).

Definition *Let $X \in C^1(\mathbb{H}^2)$ and let $A \Subset \mathbb{H}^2$ be an open set. We say that X has a stable critical point in A if there exists $r > 0$ such that any function $G \in C^1(\overline{A})$ satisfying $\|G - X\|_{C^1(\overline{A})} < r$ has a critical point in A .*

Sufficient conditions to have the existence of a stable critical point $z \in A$ for X are easily given via elementary calculus. For instance, one can assume that one of the following conditions holds:

- i) $\nabla X(z) \neq 0$ for any $z \in \partial A$, and $\deg(\nabla X, A, 0) \neq 0$, where "deg" is Browder's topological degree;
- ii) $\min_{\partial A} X > \min_A X$ or $\max_{\partial A} X < \max_A X$;
- iii) X is of class C^2 on A , it has a critical point $z_0 \in A$, and the Hessian matrix of X at z_0 is invertible.

We are in position to state our main result.

Theorem 1.1 *Let $k > 1$ and $K \in C^1(\mathbb{H}^2)$ be given. Assume that F_k^K has a stable critical point in an open set $A \Subset \mathbb{H}^2$. Then for every $\varepsilon \in \mathbb{R}$ close enough to 0, there exists an embedded $(k + \varepsilon K)$ -loop u^ε .*

Moreover, any sequence $\varepsilon_h \rightarrow 0$ has a subsequence ε_{h_j} such that $u^{\varepsilon_{h_j}} \rightarrow \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ as $j \rightarrow \infty$, where $z_0 \in \overline{A}$ is a critical point for F_k^K . In particular, if a point $z_0 \in A$ is the unique critical point for F_k^K in \overline{A} , then $u^\varepsilon \rightarrow \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ as $\varepsilon \rightarrow 0$.

Any stable critical point of the perturbation term K gives rise to a stable critical point for F_k^K , at least for k large enough. This is in essence the argument we use in Theorem 4.3 to obtain, via Theorem 1.1, the existence of $k + \varepsilon K$ -loops whenever the perturbation curvature K admits stable critical points.

The proof of Theorem 1.1 is based on a Lyapunov-Schmidt reduction technique combined with variational arguments, as proposed in [1] (see also [3, Chapter 2]).

In fact, $(k + \varepsilon K)$ -loops correspond to critical points of an energy functional $E_{k+\varepsilon K}(u) = E_k(u) + \varepsilon \int K(u) du$, where u runs in the class of nonconstant curves in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ (see Section 2.1 for details). In particular, critical points of the unperturbed functional E_k are circles of radius ρ_k . Let $\mathcal{S} = \{\omega_z \circ \xi\}$, where ξ is a rotation of \mathbb{S}^1 , $z \in \mathbb{H}^2$, and ω_z is our reference parameterization of $\partial D_{\rho_k}^{\mathbb{H}}(z)$. Clearly \mathcal{S} is a smooth three-dimensional manifold of solutions to the unperturbed problem $E'_k(u) = 0$.

The crucial and technically difficult nondegeneracy result is proved in Lemma 3.3, via an efficient functional change inspired by [17]. It states that for any $z \in \mathbb{H}^2$, the tangent space to \mathcal{S} at ω_z coincides with the set of solutions to the linear problem

$E_k''(\omega_z)\varphi = 0$. In the last section we carry out the dimensional reduction argument and complete the proof of Theorem 1.1.

We conclude the paper with a short appendix about the much more easy problem of finding loops in \mathbb{R}^2 having prescribed, almost constant curvature.

The Lyapunov-Schmidt reduction argument has been successfully used to study related geometrical problems. We limit ourselves to cite the pioneering paper [24] by R. Ye, [2, 6, 7, 8, 10, 11, 16, 17] and references therein.

2 Notation and preliminaries

The Euclidean space \mathbb{R}^2 is endowed with the scalar product $p \cdot q$ and norm $|\cdot|$, so that the disk of radius R centered at $p \in \mathbb{R}^2$ is $D_R(p) = \{z \in \mathbb{R}^2 \mid |z - p| < R\}$. The canonical basis of \mathbb{R}^2 is $e_1 = (1, 0)$, $e_2 = (0, 1)$.

Let $A, \Omega \subseteq \mathbb{R}^2$ be open sets. We write $A \Subset \Omega$ if \overline{A} is a compact subset of Ω .

We will often use complex notation for points in \mathbb{R}^2 . In particular we write $iz = (-z_2, z_1)$ and $z^2 = (z_1^2 - z_2^2, 2z_1z_2)$ for $z = (z_1, z_2) \in \mathbb{R}^2$.

Let \mathbb{S}^1 be the unit circle in the complex plane. Any $\xi \in \mathbb{S}^1$ is identified with the rotation $x \mapsto \xi x$.

The Poincaré half-plane model

We adopt as model for the two dimensional hyperbolic space the half-plane

$$\mathbb{H}^2 = \{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_2 > 0\}$$

endowed with the Riemannian metric $g_{ij}(z) = z_2^{-2}\delta_{ij}$. With some abuse of notation, we use the symbol \mathbb{H}^2 to denote the Euclidean upper half space as well.

The hyperbolic distance $d_{\mathbb{H}}(p, q)$ in \mathbb{H}^2 is related to the Euclidean one by

$$\cosh d_{\mathbb{H}}(p, q) = 1 + \frac{|p - q|^2}{2p_2q_2},$$

and the hyperbolic disk $D_{\rho}^{\mathbb{H}}(p)$ centered at $p = (p_1, p_2)$ is the Euclidean disk of center $(p_1, p_2 \cosh \rho)$ and radius $p_2 \sinh \rho$.

A *loop* in the 2-dimensional hyperbolic space \mathbb{H}^2 is a curve $u : \mathbb{S}^1 \rightarrow \mathbb{H}^2$ of class C^2 having nonzero derivative at each point. We say that u is embedded if it is injective.

If $G : \mathbb{H}^2 \rightarrow \mathbb{R}$ is a differentiable function, then $\nabla^{\mathbb{H}} G(z) = z_2^2 \nabla G(z)$, where $\nabla^{\mathbb{H}}$, ∇ are the hyperbolic and the Euclidean gradients, respectively. In particular, $\nabla^{\mathbb{H}} G(z) = 0$ if and only if $\nabla G(z) = 0$.

The hyperbolic volume form $dV_{\mathbb{H}}$ is related to the Euclidean one by $dV_{\mathbb{H}} = z_2^{-2} dz$.

The Levi-Civita connection in \mathbb{H}^2 along a curve u in Σ is given by

$$\nabla_{u'}^{\mathbb{H}} u' = u'' - u_2^{-1} \Gamma(u'), \quad (2.1)$$

where, in complex notation, $\Gamma(z) = -iz^2$. In coordinates we have

$$\Gamma(z) := (2z_1 z_2, z_2^2 - z_1^2) = z_2 z - z_1 iz, \quad \Gamma : \mathbb{H}^2 \rightarrow \mathbb{R}^2. \quad (2.2)$$

For future convenience we compute the differential

$$\Gamma'(z)w = 2(w_2 z - w_1 iz), \quad z \in \mathbb{H}^2, \quad w \in \mathbb{R}^2. \quad (2.3)$$

Isometries in \mathbb{H}^2

Hyperbolic translations are obtained by composing a horizontal (Euclidean) translation $w \mapsto w + se_1$, $s \in \mathbb{R}$ (sometimes called *parabolic isometry*), with an Euclidean homothety $w \mapsto tw$, $t > 0$ (in some literature, only homotheties are called hyperbolic translations). We obtain the two dimensional group of isometries $\mathbb{H}^2 \rightarrow \mathbb{H}^2$,

$$u \mapsto u_z := z_1 e_1 + z_2 u, \quad z \in \mathbb{H}^2.$$

Function spaces

Let $m \geq 0$, $n \geq 1$ be integer numbers. We endow $C^m(\mathbb{S}^1, \mathbb{R}^n)$ with the standard Banach space structure. If $f \in C^1(\mathbb{S}^1, \mathbb{R}^n)$, we identify $f'(x) \equiv f'(x)(ix)$, so that $f' : \mathbb{S}^1 \rightarrow \mathbb{R}^n$.

In $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^2)$ we take the Hilbertian norm

$$\|u\|_{L^2}^2 = \oint_{\mathbb{S}^1} |u(x)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{S}^1} |u(x)|^2 dx.$$

If $T \subseteq C^0(\mathbb{S}^1, \mathbb{R}^2)$, the orthogonal to T with respect to the L^2 scalar product is

$$T^\perp = \{ \varphi \in C^0(\mathbb{S}^1, \mathbb{R}^2) \mid \oint_{\mathbb{S}^1} u \cdot \varphi dx = 0 \text{ for any } u \in T \}.$$

We look at $C^m(\mathbb{S}^1, \mathbb{H}^2)$ as an open subset of the Banach space $C^m(\mathbb{S}^1, \mathbb{R}^2)$, and identify \mathbb{H}^2 with the set of constant functions in $C^m(\mathbb{S}^1, \mathbb{H}^2)$. Thus $C^m(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ contains only nonconstant curves.

2.1 The variational approach

We put

$$L(u) := L_{\mathbb{H}^2}(u) = \left(\oint_{\mathbb{S}^1} u_2^{-2} |u'|^2 dx \right)^{\frac{1}{2}}, \quad L : C^2(\mathbb{S}^1, \mathbb{H}^2) \rightarrow \mathbb{R},$$

that is a C^∞ functional, with Fréchet differential

$$L'(u)\varphi = \frac{1}{L(u)} \oint_{\mathbb{S}^1} u_2^{-2} (-u'' + u_2^{-1}\Gamma(u')) \cdot \varphi dx, \quad \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \quad (2.4)$$

When $\Sigma = \mathbb{H}^2$, problem (1.1) reads

$$u'' - u_2^{-1}\Gamma(u') = L(u)K(u)iu'. \quad (\mathcal{P}_K)$$

The system (\mathcal{P}_K) admits a variational formulation. More precisely, its nonconstant solutions are critical points of the energy functional of the form

$$E_K(u) = L(u) + A_K(u), \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2,$$

where $A_K(u)$ gives, roughly speaking, the signed area enclosed by the curve u with respect to the weight K (see Remark 2.2 below). More precisely, to introduce $A_K(u)$ we take any vectorfield $Q_K \in C^1(\mathbb{H}^2, \mathbb{R}^2)$ such that

$$\operatorname{div} Q_K(z) = z_2^{-2} K(z), \quad z \in \mathbb{H}^2$$

(here "div" is the usual Euclidean divergence). A possible choice is

$$Q_K(z_1, z_2) = \left(\frac{1}{2} z_2^{-2} \int_0^{z_1} K(t, z_2) dt \right) e_1 + \left(\frac{1}{2} \int_1^{z_2} t^{-2} K(z_1, t) dt \right) e_2.$$

Then we define

$$A_K(u) = \oint_{\mathbb{S}^1} Q_K(u) \cdot iu' dx, \quad A_K : C^2(\mathbb{S}^1, \mathbb{H}^2) \rightarrow \mathbb{R}.$$

By direct computations one gets that the functional A_K is Fréchet differentiable at any $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, with differential

$$A'_K(u)\varphi = \oint_{\mathbb{S}^1} u_2^{-2} K(u)\varphi \cdot iu' dx. \quad \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2), \quad (2.5)$$

It follows that $A_K(u)$ does not depend on the choice of the vectorfield Q_K . Further, if $K \in C^1(\mathbb{H}^2)$ then the area functional A_K is of class C^2 on $C^2(\mathbb{S}^1, \mathbb{R}^2)$.

In conclusion, the following lemma holds.

Lemma 2.1 *Let $K \in C^1(\mathbb{H}^2)$. The functional $E_K(u) = L(u) + A_K(u)$ is of class C^2 on $C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$, and*

$$L(u)E'_K(u)\varphi = \oint_{\mathbb{S}^1} u_2^{-2}(-u'' + u_2^{-1}\Gamma(u') + L(u)K(u)iu') \cdot \varphi \, dx$$

for any $u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$, $\varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2)$. In particular, if $u_0 \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ is a critical point for the functional $E_K(u)$, then u_0 solves (\mathcal{P}_K) , hence it is an hyperbolic K -loop.

Remark 2.2 *Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be an embedded loop. Then u is a regular parameterization of the boundary of an open set $\Omega_u \Subset \mathbb{H}^2$. Assume for instance that u is positively oriented, so that iu' gives the inner direction to Ω_u . Then*

$$A_K(u) = -\frac{1}{2\pi} \int_{\partial\Omega} Q_K(z) \cdot \nu \, ds = -\frac{1}{2\pi} \int_{\Omega} K(z) dV_{\mathbb{H}}$$

by the divergence theorem.

2.2 Nonexistence results

We start with a simple result that should be well known. We sketch its proof by adapting the argument in [15, p. 194].

Proposition 2.3 *Let $K \in C^0(\mathbb{H}^2)$. If $\|K\|_{\infty} \leq 1$ then no K -loop exists.*

Proof. Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be a K -loop. We need to show that $|K| > 1$ somewhere in \mathbb{H}^2 . Take the smallest closed disk $D_{\rho} = \overline{D_{\rho}^{\mathbb{H}}(z)}$ containing $u(\mathbb{S}^1)$. Then ∂D_{ρ} is tangent to $u(\mathbb{S}^1)$ at some point. At the contact point the absolute value of the curvature of u can not be smaller than the curvature $1/\tanh \rho$ of the circle ∂D_{ρ} , use a local comparison principle. The conclusion readily follows from $\tanh \rho < 1$. \square

Next, we point out few necessary conditions for the existence of K -loops.

Lemma 2.4 *Let $K \in C^1(\mathbb{H}^2)$ and let $\Omega \subset \mathbb{H}^2$ be a bounded open domain. Assume that $\partial\Omega$ is parameterized by a K -loop $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$. Then*

$$\int_{\Omega} \nabla K(z) \cdot e_1 \, dV_{\mathbb{H}^2} = \int_{\Omega} \nabla K(z) \cdot z \, dV_{\mathbb{H}^2} = \int_{\Omega} \nabla K(z) \cdot z^2 \, dV_{\mathbb{H}^2} = 0.$$

Proof. Direct computations based on integration by parts give

$$L'(u)e_1 = L'(u)u = L'(u)i(\Gamma u) = 0, \quad (2.6)$$

see (2.4) and (2.2). In addition, the curve u solves

$$-L(u)L'(u)\varphi = \oint_{\mathbb{S}^1} u_2^{-2}K(u)\varphi \cdot iu' dx \quad \text{for any } \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2).$$

Since $iu'(x) \neq 0$ is parallel to the outer normal ν to Ω at $u(x) \in \partial\Omega$, we infer that

$$\int_{\partial\Omega} z_2^{-2}K(z)e_1 \cdot \nu = \int_{\partial\Omega} z_2^{-2}K(z)z \cdot \nu = \int_{\partial\Omega} z_2^{-2}K(z)i\Gamma(z) \cdot \nu = 0.$$

Recall that we identify $i\Gamma(z) = z^2$, then use the divergence theorem to get

$$\int_{\Omega} \operatorname{div}(z_2^{-2}K(z)e_1) dz = \int_{\Omega} \operatorname{div}(z_2^{-2}K(z)z) dz = \int_{\Omega} \operatorname{div}(z_2^{-2}K(z)z^2) dz = 0.$$

The conclusion readily follows. □

Remark 2.5 *The identities in (2.6) hold indeed for any curve u , and are related to the group of isometries in \mathbb{H}^2 . Notice indeed that $z \mapsto e_1, z \mapsto z, z \mapsto z^2$ are infinitesimal Killing vectorfields in \mathbb{H}^2 .*

Lemma 2.4 readily implies the next nonexistence result.

Corollary 2.6 *Let $K \in C^1(\mathbb{H}^2)$ be a given curvature function. Assume that one of the following conditions hold,*

- i) K is strictly monotone in the e_1 direction;*
- ii) K is radially strictly monotone, that is, $\nabla K(z) \cdot z$ never vanishes on \mathbb{H}^2 ;*
- iii) $\nabla K(z) \cdot z^2$ never vanishes on \mathbb{H}^2*

Then no embedded K -loop exists.

3 The unperturbed problem

In this section we take a constant $k > 1$ and study the system

$$u'' - u_2^{-1} \Gamma(u') = L(u) k i u'. \quad (\mathcal{P}_k)$$

We start by introducing the radius

$$R_k := \sinh \rho_k = \frac{1}{k} \cosh \rho_k = \frac{1}{\sqrt{k^2 - 1}}$$

and the reference loop $\omega : \mathbb{S}^1 \rightarrow \mathbb{H}^2$,

$$\omega(x) = \frac{1}{k - x_2} \left(x_1, \frac{1}{R_k} \right), \quad x = x_1 + i x_2 \in \mathbb{S}^1. \quad (3.1)$$

Notice that

$$|\omega - k R_k e_2| = R_k, \quad (3.2)$$

hence ω is a (positive) parametrization of the Euclidean circle $\partial D_{R_k}(k R_k e_2)$, that coincides with the hyperbolic circle $\partial D_{\rho_k}^{\mathbb{H}}(e_2)$. The next identities will be very useful:

$$\omega' = \omega_2 i(\omega - k R_k e_2) \quad (3.3)$$

$$\omega_2^{-1} \Gamma(\omega') = (\omega_2 - k R_k) i \omega' + \omega_1 \omega' \quad (3.4)$$

$$\omega_2^{-1} |\omega'| \equiv L(\omega) = R_k. \quad (3.5)$$

By differentiating (3.3) and using (3.5) one easily gets that ω solves (\mathcal{P}_k) . Next, for $z = (z_1, z_2) \in \mathbb{H}^2$ we parameterize $\partial D_{\rho_k}^{\mathbb{H}}(z)$ by the function

$$\omega_z = z_1 e_1 + z_2 \omega.$$

Notice that $\omega = \omega_{e_2}$. It is easy to check that for any rotation $\xi \in \mathbb{S}^1$ and any point $z \in \mathbb{H}^2$, the circle $\omega_z \circ \xi$ solves (\mathcal{P}_k) as well. Further, by Remark 2.2 we have

$$F_k^K(z) := \int_{D_{\rho_k}^{\mathbb{H}}(z)} K(z) dV_{\mathbb{H}} = -2\pi A_K(\omega_z). \quad (3.6)$$

We know that any nonconstant solution u to (\mathcal{P}_k) has constant curvature k , hence is a circle of hyperbolic radius ρ_k . Actually we need a sharper uniqueness result, that is, we have to classify solutions to (\mathcal{P}_k) .

Lemma 3.1 *Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be a nonconstant solution to (\mathcal{P}_k) . Then $\mu := L(u)/L(\omega)$ is an integer number, and there exist $\xi \in \mathbb{S}^1$, $z = (z_1, z_2) \in \mathbb{H}^2$ such that $u(x) = \omega_z \circ \xi$. In particular, u parameterizes $\partial D_{\rho_k}(z)$, and $L(u) = \mu L(\omega) = \mu R_k$.*

Proof. We have

$$\omega_2(-i) = e^{-\rho_k} = \min_{x \in \mathbb{S}^1} \omega_2(x) \ , \quad \omega(-i) = e^{-\rho_k} e_2 \ , \quad \omega'(-i) = e^{-\rho_k} L(\omega) e_1 \ .$$

Let $x_u \in \mathbb{S}^1$ such that

$$u_2(x_u) = m_u := \min_{x \in \mathbb{S}^1} u_2(x) \ .$$

Now we show that

$$u'(x_u) = m_u L(u) e_1. \quad (3.7)$$

Clearly $u'_2(x_u) = 0$ and $u''_2(x_u) \geq 0$. We first infer that $\Gamma(u'(x_u)) = -u'_1(x_u) i u'(x_u)$, compare with (2.2). Thus the system (\mathcal{P}_k) for the second coordinate gives

$$(L(u)k - m_u^{-1} u'_1(x_u)) u'_1(x_u) = u''_2(x_u) \geq 0,$$

that implies $u'_1(x_u) \geq 0$. On the other hand, $u_2^{-1}|u'| \equiv L(u)$ on \mathbb{S}^1 . Thus $u'_1(x_u) = |u'(x_u)| = m_u L(u)$, and (3.7) is proved.

In particular, u solves the Cauchy problem

$$v'' = v_2^{-1} \Gamma(v') + k L(u) i v' \ , \quad v(x_u) = u(x_u) \ , \quad v'(x_u) = m_u L(u) e_1. \quad (3.8)$$

It is easy to check that the function

$$\tilde{u}(x) := m_u e^{\rho_k} \omega(-ix_u^{-\mu} x^\mu) + u_1(x_u) e_1$$

solves (3.8) as well (use $f'(x) = i\mu x^\mu$ for $f(x) = x^\mu$, $f : \mathbb{S}^1 \rightarrow \mathbb{C}$). Thus $\tilde{u}(x) = u(x)$ for any $x \in \mathbb{S}^1$ and hence $u(x) = \omega_z \circ \xi$, where $z_1 = u_1(x_u)$, $z_2 = m_u e^{\rho_k}$, $\xi = -ix_u^{-\mu}$. Finally, μ is an integer number because u and ω are both well defined on \mathbb{S}^1 . \square

The linearized problem

By Lemma 3.1, the 3-dimensional manifold

$$\mathcal{S} = \{ \omega_z \circ \xi \mid \xi \in \mathbb{S}^1 \ , \ z \in \mathbb{H}^2 \} \subset C^2(\mathbb{S}^1, \mathbb{H}^2), \quad \omega_z = z_1 e_1 + z_2 \omega$$

is the set of embedded solutions to (\mathcal{P}_k) . The tangent space to \mathcal{S} at ω_z is

$$T_{\omega_z} \mathcal{S} = T_\omega \mathcal{S} = \langle \omega', e_1, \omega \rangle.$$

Every loop in $\omega_z \circ \xi \in \mathcal{S}$ is a critical point for the energy functional

$$E_k(u) = L(u) + A_k(u) = \left(\int_{\mathbb{S}^1} u_2^{-2} |u'|^2 dx \right)^{\frac{1}{2}} - k \int_{\mathbb{S}^1} u_2^{-1} u'_1 dx$$

on $C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus H^2$, and $E_k(\omega_z \circ \xi) = E_k(\omega)$ is a constant. More generally one has

$$E_k(z_1 e_1 + z_2 u \circ \xi) = E_k(u) \quad \text{for any } \xi \in \mathbb{S}^1, z \in \mathbb{H}^2. \quad (3.9)$$

In order to handle the differential of E_k , it is convenient to introduce the function $J_0 : C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2 \rightarrow C^0(\mathbb{S}^1, \mathbb{H}^2)$ given by

$$\begin{aligned} J_0(u) &= -(u_2^{-2}u')' - u_2^{-3}|u'|^2 e_2 + L(u)ku_2^{-2}iu' \\ &= u_2^{-2}(-u'' + u_2^{-1}\Gamma(u') + L(u)k iu'). \end{aligned} \quad (3.10)$$

By Lemma 2.1 we have

$$L(u)E'_k(u)\varphi = \oint_{\mathbb{S}^1} J_0(u) \cdot \varphi \, dx \quad \text{for any } \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \quad (3.11)$$

By differentiating (3.9) at $\xi = 1$, $z = e_2$ we readily get $E'_k(u)u' = E'_k(u)e_1 = E'_k(u)u = 0$ for any nonconstant curve $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, that is,

$$\oint_{\mathbb{S}^1} J_0(u) \cdot u' \, dx = 0, \quad \oint_{\mathbb{S}^1} J_0(u) \cdot e_1 \, dx = 0, \quad \oint_{\mathbb{S}^1} J_0(u) \cdot u \, dx = 0. \quad (3.12)$$

Now we differentiate (3.11) with respect to u , at $u = \omega_z$. From $E'_k(\omega_z) = 0$ we get

$$L(\omega)E''_k(\omega_z)[\varphi, \tilde{\varphi}] = \oint_{\mathbb{S}^1} J'_0(\omega_z)\varphi \cdot \tilde{\varphi} \, dx \quad \text{for any } \varphi, \tilde{\varphi} \in C^2(\mathbb{S}^1, \mathbb{R}^2).$$

Since E_k is of class C^2 , then $J'_0(\omega_z)$ is self-adjoint in L^2 , that means

$$\oint_{\mathbb{S}^1} J'_0(\omega_z)\varphi \cdot \tilde{\varphi} \, dx = \oint_{\mathbb{S}^1} J'_0(\omega_z)\tilde{\varphi} \cdot \varphi \, dx \quad \text{for any } \varphi, \tilde{\varphi} \in C^2(\mathbb{S}^1, \mathbb{R}^2). \quad (3.13)$$

Finally, we differentiate $E'_k(\omega_z \circ \xi) = 0$ with respect to the variables $\xi \in \mathbb{S}^1, z \in \mathbb{H}^2$ to get $T_{\omega_z}\mathcal{S} \subseteq \ker J'_0(\omega_z)$. We shall see in the crucial Lemma 3.3 below that indeed $T_{\omega_z}\mathcal{S} = \ker J'_0(\omega_z)$.

This will be done via a useful functional change.

A functional change and nondegeneracy

In order to avoid tricky computations, we use in $C^m(\mathbb{S}^1, \mathbb{R}^2)$, $m \geq 0$, the orthogonal frame $\omega', i\omega'$. We introduce the isomorphism

$$\Phi(g) = g_1\omega' + g_2i\omega', \quad \Phi : C^m(\mathbb{S}^1, \mathbb{R}^2) \rightarrow C^m(\mathbb{S}^1, \mathbb{R}^2)$$

together with its inverse $\Phi^{-1}(\varphi) = R_k^{-2}\omega_2^{-2}(\varphi \cdot \omega' e_1 + \varphi \cdot i\omega' e_2)$ (recall that $|\omega'| = R_k\omega_2$) and the differential operator

$$Bg = -g'' - kR_k i g' + R_k^2(g_2 - k^2 \oint_{\mathbb{S}^1} g_2 dx) e_2, \quad g \in C^2(\mathbb{S}^1, \mathbb{R}^2). \quad (3.14)$$

Lemma 3.2 *Let z be any point in \mathbb{H}^2 . The following facts hold.*

- i) $J'_0(\omega_z)(\Phi(g)) = z_2^{-2}\omega_2^{-2}\Phi(Bg)$ for any $g \in C^2(\mathbb{S}^1, \mathbb{S}^2)$;
- ii) $\oint_{\mathbb{S}^1} \omega_2^{-2}\Phi(g) \cdot \Phi(\tilde{g}) dx = R_k^2 \oint_{\mathbb{S}^1} g \cdot \tilde{g} dx$ for any $g, \tilde{g} \in C^2(\mathbb{S}^1, \mathbb{S}^2)$

Proof. Since $J_0(\omega_z) = z_2^{-1}J_0(\omega) = 0$ and $J'_0(\omega_z) = z_2^{-2}J'_0(\omega)$, it suffices to prove i) for $z = e_2$, that corresponds to $\omega_z = \omega$. We have to show that

$$\mathcal{J}(\varphi) := \omega_2^2 J'_0(\omega)\varphi = \Phi(Bg), \quad \text{where } \varphi = g_1\omega' + g_2 i\omega'. \quad (3.15)$$

To compute $\mathcal{J}(\varphi)$ it is convenient to recall (3.10) and to differentiate the identity

$$u_2^2 J_0(u) = -u'' + u_2^{-1}\Gamma(u') + L(u)k iu'$$

at $u = \omega$. Since $J_0(\omega) = 0$ and $L(\omega) = R_k$, we get

$$\mathcal{J}(\varphi) = -\varphi'' + kR_k i\varphi' + \omega_2^{-1}\Gamma'(\omega')\varphi' - \omega_2^{-2}\varphi_2\Gamma(\omega') + k(L'(\omega)\varphi)i\omega'.$$

From (2.3) we find $\Gamma'(\omega')\varphi' = 2\varphi'_2\omega' - 2\varphi'_1 i\omega'$. Taking also (3.4) into account, we obtain

$$\mathcal{J}(\varphi) = -\varphi'' + kR_k i\varphi' + A_1(\varphi)\omega' - (A_2(\varphi) - kL'(\omega)\varphi)i\omega',$$

where

$$A_1(\varphi) = (2\varphi'_2 - \varphi_2\omega_1)\omega_2^{-1}, \quad A_2(\varphi) = (2\varphi'_1 + \varphi_2(\omega_2 - kR_k))\omega_2^{-1}.$$

To compute the differential $L'(\omega)$ at φ we recall that ω solves (\mathcal{P}_k) . Thus (2.4) gives

$$L'(\omega)\varphi = -k \oint_{\mathbb{S}^1} \omega_2^{-2}\varphi \cdot i\omega' dx.$$

For the next computations we observe that the loop ω solves several useful differential systems. In particular, from (\mathcal{P}_k) , (3.4), (3.5) and (3.3) it follows that

$$\omega'' = \omega_1\omega' + \omega_2 i\omega', \quad \omega''' = (\omega_1^2 - 2\omega_2 + kR_k\omega_2)\omega' + 3\omega_1\omega_2 i\omega'. \quad (3.16)$$

Now we take any $\psi \in C^2(\mathbb{S}^1, \mathbb{R})$ and we look for an explicit formula for $\mathcal{J}(\psi\omega')$. Clearly $L'(\omega)(\psi\omega') = 0$, as $\omega' \cdot i\omega' \equiv 0$. Direct computations based on (3.16) give

$$\begin{aligned} -(\psi\omega')'' + kR_k i(\psi\omega')' &= (-\psi'' - 2\omega_1\psi' - (\omega_1^2 - 2\omega_2^2 + 2kR_k\omega_2)\psi)\omega' \\ &\quad + ((kR_k - 2\omega_2)\psi' + (kR_k - 2\omega_2)\omega_1\psi)i\omega' \end{aligned}$$

$$\begin{aligned} A_1(\psi\omega') &= 2\omega_1\psi' - (2\omega_2^2 - 2kR_k\omega_2 - \omega_1^2)\psi \\ A_2(\psi\omega') &= 2(kR_k - \omega_2)\psi' - (kR_k - 3\omega_2)\psi, \end{aligned}$$

and we find the formula

$$\mathcal{J}(\psi \omega') = -\psi'' \omega' - k R_k \psi' i \omega'. \quad (3.17)$$

Now we handle $\mathcal{J}(\psi i \omega')$. From (3.5) we get

$$k L'(\omega)(\psi i \omega') = -k^2 \oint_{\mathbb{S}^1} \omega_2^{-2} |\omega'|^2 \psi dx = -k^2 R_k^2 \oint_{\mathbb{S}^1} \psi dx.$$

Then we use (3.2–3.5) and (3.16) to compute

$$\begin{aligned} -(\psi i \omega')'' + k R_k i(\psi i \omega')' &= ((2\omega_2 - k R_k) \psi' + (3\omega_2 - k R_k) \omega_1 \psi) \omega' \\ &\quad + (-\psi'' - 2\omega_1 \psi' - (\omega_1^2 - 2\omega_2^2 + 2k R_k \omega_2) \psi) i \omega' \end{aligned}$$

$$\begin{aligned} A_1(\psi i \omega') &= -2(\omega_2 - k R_k) \psi' - (3\omega_2 - k R_k) \omega_1 \psi \\ A_2(\psi i \omega') &= -2\omega_1 \psi' + (\omega_2^2 - k^2 R_k^2 - 2\omega_1^2) \psi. \end{aligned}$$

Since $R_k^2 = |\omega - k R_k e_2|^2 = |\omega|^2 - 2k R_k \omega_2 + k^2 R_k^2$ by (3.2), we arrive at

$$\mathcal{J}(\psi i \omega') = k R_k \psi' \omega' + (-\psi'' + R_k^2 \psi - k^2 R_k^2 \oint_{\mathbb{S}^1} \psi dx) i \omega',$$

that together with (3.17) gives

$$\begin{aligned} \mathcal{J}(g_1 \omega' + g_2 i \omega') &= (-g_1'' - k R_k g_2') \omega' \\ &\quad + (-g_2'' + k R_k g_1' + R_k^2 g_2 - k^2 R_k^2 \oint_{\mathbb{S}^1} g_2 dx) i \omega' \end{aligned}$$

and concludes the proof of (3.15). The proof of *i*) is complete; the formula in *ii*) is immediate, because $\omega' \cdot i \omega' \equiv 0$ and $|\omega'| = R_k \omega_2$. \square

We are in position to prove the main result of this section.

Lemma 3.3 (Nondegeneracy) *Let z be any point in \mathbb{H}^2 . The following facts hold.*

- i) $\ker J'(\omega_z) = T_\omega \mathcal{S}$;*
- ii) If $J'_0(\omega_z) \varphi \in T_\omega \mathcal{S}$, then $\varphi \in T_\omega \mathcal{S}$;*
- iii) For any $u \in T_\omega \mathcal{S}^\perp$ there exists a unique $\varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2) \cap T_\omega \mathcal{S}^\perp$ such that $J'_0(\omega_z) \varphi = u$.*

Proof. We start by studying the kernel of the operator B in (3.14). In coordinates, the linear problem $Bg = 0$ becomes

$$-g_1'' + k R_k g_2' = 0, \quad -g_2'' - k R_k g_1' + R_k^2 \left(g_2 - k^2 \oint_{\mathbb{S}^1} g_2 dx \right) = 0,$$

that is clearly equivalent to

$$\oint_{\mathbb{S}^1} g_2 dx = 0, \quad -g_1'' + kR_k g_2' = 0, \quad -g_2'' - kR_k g_1' + R_k^2 g_2 = 0 \quad (3.18)$$

because $k > 1$. The system (3.18) can be studied via elementary techniques. The conclusion is that $\ker B = \langle e_1, \gamma, \gamma' \rangle$, where $\gamma = \frac{1}{R_k} (kR_k x_1, -x_2)$. Since $\Phi(e_1) = \omega'$, $\Phi(\gamma) = \omega$ and $\Phi(\gamma') = e_1 - \omega'$, thanks to Lemma 3.2 we have

$$\ker J'_0(\omega_z) = \Phi(\ker B) = \Phi(\langle e_1, \gamma, \gamma' \rangle) = T_\omega \mathcal{S},$$

and the first claim is proved.

Now we prove *ii*). If $\tau := J'_0(\omega_z)\varphi \in T_\omega \mathcal{S} = \ker J'(\omega_z)$, then $J'_0(\omega_z)\tau = 0$. Taking (3.13) into account, we obtain

$$\oint_{\mathbb{S}^1} |J'_0(\omega_z)\varphi|^2 dx = \oint_{\mathbb{S}^1} J'_0(\omega_z)\varphi \cdot \tau dx = \oint_{\mathbb{S}^1} J'_0(\omega_z)\tau \cdot \varphi dx = 0.$$

Thus $J'_0(\omega_z)\varphi = 0$, that means $\varphi \in T_\omega \mathcal{S}$.

It remains to prove *iii*). If $u \in T_\omega \mathcal{S}^\perp$, then $\Phi^{-1}(\omega_z^2 u)$ is orthogonal to $\ker B$ by *ii*) in Lemma 3.2. One can compute the Fourier coefficients of the unique solution $g_u \in \ker B^\perp$ of the system $Bg_u = \Phi^{-1}(\omega_z^2 u)$. Then $J'_0(\omega)(z_2^2 \Phi(g_u)) = u$ by *i*) in Lemma 3.2. The function φ defined as the L^2 -projection of $z_2^2 \Phi(g_u)$ on $T_\omega \mathcal{S}^\perp$ solves $J'_0(\omega)\varphi = u$ as well, and is uniquely determined by u .

The lemma is completely proved. \square

4 The perturbed problem

Let $k > 1$, $K \in C^1(\mathbb{H}^2)$ be given, and let $\varepsilon \in \mathbb{R}$ be a varying parameter. In this section we study the system

$$u'' - u_2^{-1} \Gamma(u') = L(u)(k + \varepsilon K(u)) iu'. \quad (\mathcal{P}_{k+\varepsilon K})$$

We start with a necessary condition for the existence of solutions to $(\mathcal{P}_{k+\varepsilon K})$ having some prescribed behavior as $\varepsilon \rightarrow 0$.

Theorem 4.1 *Let $k > 1$, $K \in C^1(\mathbb{H}^2)$, and $\varepsilon_h \rightarrow 0$ be given. For any integer h , let $u_h \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ be a solution to*

$$u_h'' = (u_h)_2^{-1} \Gamma(u_h') + L(u_h)(k + \varepsilon_h K(u_h)) iu_h', \quad (\mathcal{P}_{\varepsilon_h})$$

and assume that

$$L(u_h) \rightarrow L_\infty > 0, \quad u_h \rightarrow U \text{ uniformly, for some } U \in C^0(\mathbb{S}^1, \mathbb{H}^2).$$

Then there exist $\mu \in \mathbb{N}$, $\xi \in \mathbb{S}^1$ and a critical point $z \in \mathbb{H}^2$ for F_k^K , such that $U(x) = \omega_z(\xi x^\mu)$.

Proof. We have $|u'_h| \equiv L(u_h)(u_h)_2$, thus the sequence $|u'_h|$ is uniformly bounded. It follows that u''_h is uniformly bounded as well, because u_h solves $(\mathcal{P}_{\varepsilon_h})$. Thus, u'_h is bounded in $C^{0,s}$ for any $s \in (0, 1)$ and using $(\mathcal{P}_{\varepsilon_h})$ again we infer that the sequence u_h converges in $C^{2,s}$ for any $s \in (0, 1)$. In particular, $U \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, $L_\infty = L(U)$ and U solves

$$U'' = U_2^{-1} \Gamma(U') + L(U) k i U'.$$

Lemma 3.1 applies and gives the existence of $\xi \in \mathbb{S}^1$, $z \in \mathbb{H}^2$, $\mu \in \mathbb{N}$ such that $U(x) = \omega_z(\xi x^\mu)$ and $L_\infty = L(U) = \mu L(\omega)$.

It remains to prove that z is a critical point for F_k^K . We rewrite $(\mathcal{P}_{\varepsilon_h})$ in the form

$$J_0(u_h) + \varepsilon_h L(u_h)(u_h)_2^{-2} K(u_h) i u'_h = 0, \quad (4.1)$$

see (3.10). Then we test (4.1) with the functions e_1 and u_h . Taking (3.12) into account, we find

$$\oint_{\mathbb{S}^1} (u_h)_2^{-2} K(u_h) e_1 \cdot i u'_h dx = 0, \quad \oint_{\mathbb{S}^1} (u_h)_2^{-2} K(u_h) u_h \cdot i u'_h dx = 0.$$

Since $u_h \rightarrow U(x) = \omega_z(\xi x^\mu)$, in the limit as $h \rightarrow \infty$ we obtain

$$\mu \oint_{\mathbb{S}^1} (\omega_z)_2^{-2} K(\omega_z) e_1 \cdot i \omega'_z dx = 0, \quad \mu \oint_{\mathbb{S}^1} (\omega_z)_2^{-2} K(\omega_z) \omega_z \cdot i \omega'_z dx = 0,$$

that is,

$$\partial_{z_1} A_K(\omega_z) = A'_K(\omega_z) e_1 = 0, \quad \partial_{z_2} A_K(\omega_z) = A'_K(\omega_z) \omega = 0.$$

Thus z is a critical point for F_k^K because of (3.6). □

4.1 Finite dimensional reduction

By Lemma 2.1, $k + \varepsilon K$ -loops are the critical points of the functional

$$E_{k+\varepsilon K}(u) = E_k(u) + \varepsilon A_K(u) = L(u) + k A_1(u) + \varepsilon A_K(u), \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2.$$

We introduce the C^1 function $J_\varepsilon : C^2(\mathbb{R}, \mathbb{H}^2) \setminus \mathbb{H}^2 \rightarrow C^0(\mathbb{R}, \mathbb{H}^2)$,

$$\begin{aligned} J_\varepsilon(u) &= J_0(u) + \varepsilon L(u) u_2^{-2} K(u) i u' \\ &= u_2^{-2} (-u'' + u_2^{-1} \Gamma(u') + L(u)(k + \varepsilon K(u)) i u'), \end{aligned}$$

compare with (3.10), so that

$$L(u) E'_{k+\varepsilon K}(u) \varphi = \oint_{\mathbb{S}^1} J_\varepsilon(u) \cdot \varphi dx, \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2), \quad \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \quad (4.2)$$

We will look for critical points for $E_{k+\varepsilon K}$ by solving the problem $J_\varepsilon(u) = 0$.

First, we notice that $E_{k+\varepsilon K}(u \circ \xi) = E_{k+\varepsilon K}(u)$ for any $\xi \in \mathbb{S}^1$, that implies

$$\oint_{\mathbb{S}^1} J_\varepsilon(u) \cdot u' dx = 0 \quad \text{for any } \varepsilon \in \mathbb{R}, u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2. \quad (4.3)$$

In the next crucial lemma we carry out the Lyapunov-Schmidt procedure, in which we take advantage of the variational structure of problem $(\mathcal{P}_{k+\varepsilon K})$.

Lemma 4.2 *Let $\Omega \Subset \mathbb{H}^2$ be a given open set. There exist $\bar{\varepsilon} > 0$ and a C^1 function*

$$[-\bar{\varepsilon}, \bar{\varepsilon}] \times \bar{\Omega} \rightarrow C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2, \quad (\varepsilon, z) \mapsto u_z^\varepsilon$$

such that the following facts hold.

- i) u_z^ε is an embedded loop and $u_z^0 = \omega_z$;
- ii) $u_z^\varepsilon - \omega_z \in T_\omega \mathcal{S}^\perp$;
- iii) $J_\varepsilon(u_z^\varepsilon) \in T_\omega \mathcal{S}$. More precisely,

$$\frac{1}{L(u_z^\varepsilon)} J_\varepsilon(u_z^\varepsilon) = \partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon)) e_1 + \left(\oint_{\mathbb{S}^1} |\omega|^2 dx \right)^{-1} \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon)) \omega; \quad (4.4)$$

- iv) As $\varepsilon \rightarrow 0$, we have

$$E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z) = o(\varepsilon) \quad (4.5)$$

uniformly on Ω , together with the derivatives with respect to the variable z .

Proof. In order to shorten formulae, for $r > 0$, $m \in \{0, 2\}$ and $\delta > 0$ we write

$$\Omega_r = \{z \in \mathbb{R}^2 \mid \text{dist}(z, \Omega) < r\}, \\ C^m = C^m(\mathbb{S}^1, \mathbb{R}^2), \quad \mathcal{U}_\delta := \{\eta \in C^2 \mid |\eta(x)| < \delta \text{ for any } x \in \mathbb{S}^1\}.$$

Take $r, \delta > 0$ small enough, so that $\bar{\Omega}_{2r} \subset \mathbb{H}^2$ and $\omega_z + \eta \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ for any $z \in \bar{\Omega}_{2r}$, $\eta \in \mathcal{U}_\delta$. Consider the differentiable function

$$\mathcal{F} : (\mathbb{R} \times \Omega_{2r}) \times \mathcal{U}_\delta \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow C^0 \times (\mathbb{R} \times \mathbb{R}^2), \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2),$$

whose coordinates

$$\mathcal{F}_1 : (\mathbb{R} \times \Omega_{2r}) \times \mathcal{U}_\delta \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow C^0, \quad \mathcal{F}_2 : (\mathbb{R} \times \Omega_{2r}) \times \mathcal{U}_\delta \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow \mathbb{R} \times \mathbb{R}^2$$

are given by

$$\begin{aligned} \mathcal{F}_1(\varepsilon, z; \eta; t, \vartheta) &= J_\varepsilon(\omega_z + \eta) - t\omega' - \vartheta_1 e_1 - \vartheta_2 \omega, \\ \mathcal{F}_2(\varepsilon, z; \eta; t, \vartheta) &= \left(\oint_{\mathbb{S}^1} \eta \cdot \omega' dx, \oint_{\mathbb{S}^1} \eta_1 dx, \oint_{\mathbb{S}^1} \eta \cdot \omega dx \right). \end{aligned}$$

Take $z \in \Omega_{2r}$ and notice that $\mathcal{F}(0, z; 0; 0, 0) = 0$ because $J_0(\omega_z) = 0$. The next goal is to solve the equation $\mathcal{F}(\varepsilon, z; \eta; t, \vartheta) = (0, 0)$ in a neighborhood of $(\varepsilon, z) = (0, z)$, $(\eta; t, \vartheta) = (0; 0, 0)$ via the implicit function theorem. Let

$$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow C^0 \times (\mathbb{R} \times \mathbb{R}^2)$$

be the differential of $\mathcal{F}(0, z; \cdot; \cdot, \cdot)$ computed at $(\eta; t, \vartheta) = (0; 0, 0) \in C^2 \times (\mathbb{R} \times \mathbb{R}^2)$. We need to prove that \mathcal{L} is invertible. Explicitly, we have

$$\mathcal{L}_1 : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow C^0, \quad \mathcal{L}_1(\varphi; a, p) = J'_0(\omega_z)\varphi - a\omega' - p_1e_1 - p_2\omega$$

$$\mathcal{L}_2 : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \rightarrow \mathbb{R} \times \mathbb{R}^2, \quad \mathcal{L}_2(\varphi; a, p) = \left(\oint_{\mathbb{S}^1} \varphi \cdot \omega' dx, \oint_{\mathbb{S}^1} \varphi_1 dx, \oint_{\mathbb{S}^1} \varphi \cdot \omega dx \right).$$

If $\mathcal{L}_1(\varphi; a, p) = 0$ then $J'_0(\omega_z)\varphi \in T_\omega\mathcal{S}$, hence $\varphi \in T_\omega\mathcal{S}$ by *ii*) in Lemma 3.3. If $\mathcal{L}_2(\varphi; a, p) = 0$ then $\varphi \in T_\omega\mathcal{S}^\perp$. Therefore, the operator \mathcal{L} is injective.

To prove surjectivity take $u \in C^0$, $(b, q) \in \mathbb{R} \times \mathbb{R}^2$. We have to find $\varphi \in C^2$, $(a, p) \in \mathbb{R} \times \mathbb{R}^2$ satisfying $\mathcal{L}_1(\varphi; a, p) = u$ and $\mathcal{L}_2(\varphi; a, p) = (b, q_1, q_2)$, that is,

$$J'_0(\omega_z)\varphi = u + a\omega' + p_1e_1 + p_2\omega \tag{4.6}$$

$$\oint_{\mathbb{S}^1} \varphi \cdot \omega' dx = b, \quad \oint_{\mathbb{S}^1} \varphi_1 dx = q_1, \quad \oint_{\mathbb{S}^1} \varphi \cdot \omega dx = q_2. \tag{4.7}$$

By (3.13), for any $\varphi \in C^2$, $\tau \in T_\omega\mathcal{S} = \langle \omega', e_1, \omega \rangle = \ker J'_0(\omega_z)$ we have

$$\oint_{\mathbb{S}^1} J'_0(\omega_z)\varphi \cdot \tau dx = \oint_{\mathbb{S}^1} J'_0(\omega_z)\tau \cdot \varphi dx = 0.$$

Thus the unknowns $a \in \mathbb{R}$ and $p = (p_1, p_2) \in \mathbb{R}^2$ are determined by the condition

$$\oint_{\mathbb{S}^1} u \cdot \tau dx + a \oint_{\mathbb{S}^1} \omega' \cdot \tau dx + p_1 \oint_{\mathbb{S}^1} e_1 \cdot \tau dx + p_2 \oint_{\mathbb{S}^1} \omega \cdot \tau dx = 0 \quad \text{for any } \tau \in T_\omega\mathcal{S}. \tag{4.8}$$

Now we look for the L^2 projection of the unknown function φ on $T_\omega\mathcal{S}$ and its L^2 projection on $T_\omega\mathcal{S}^\perp$. The tangential component $\varphi^\top \in T_\omega\mathcal{S} = \langle \omega', e_1, \omega \rangle$ is uniquely determined by (4.7). Next, we notice that $u + a\omega' + p_1e_1 + p_2\omega \in T_\omega\mathcal{S}^\perp$ by (4.8); then we use *iii*) in Lemma 3.3 to find $\varphi^\perp \in C^2 \cap T_\omega\mathcal{S}^\perp$ such that

$$J'_0(\omega_z)\varphi^\perp = u + a\omega' + p_1e_1 + p_2\omega.$$

The function $\varphi = \varphi^\top + \varphi^\perp$ solves (4.6) because $J'_0(\omega_z)\varphi = J'_0(\omega_z)\varphi^\perp$, and surjectivity is proved.

We can now apply the implicit function theorem for any fixed $z \in \Omega_{2r}$. Actually, thanks a compactness argument we have that there exist $\varepsilon' > 0$ and (uniquely determined) C^1 functions

$$\begin{aligned} \eta : (-\varepsilon', \varepsilon') \times \Omega_r &\rightarrow \mathcal{U}_\delta \subset C^2 & t : (-\varepsilon', \varepsilon') \times \Omega_r &\rightarrow \mathbb{R} & \vartheta : (-\varepsilon', \varepsilon') \times \Omega_r &\rightarrow \mathbb{R}^2 \\ \eta : (\varepsilon, z) &\mapsto \eta^\varepsilon(z) & t : (\varepsilon, z) &\mapsto t^\varepsilon(z), & \vartheta : (\varepsilon, z) &\mapsto \vartheta^\varepsilon(z) \end{aligned}$$

such that

$$\eta^0(z) = 0, \quad t^0(z) = 0, \quad \vartheta^0(z) = 0, \quad \mathcal{F}(\varepsilon, z; \eta^\varepsilon(z); t^\varepsilon(z), \vartheta^\varepsilon(z)) = 0.$$

We introduce the C^1 function

$$(-\varepsilon', \varepsilon') \times \Omega_r \rightarrow C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2, \quad (\varepsilon, z) \mapsto u_z^\varepsilon := \omega_z + \eta^\varepsilon(z),$$

that clearly satisfies $u_z^0 = \omega_z$. Since ω_z is embedded, then u_z^ε is embedded as well, provided that ε' is small enough. Moreover we have

$$J_\varepsilon(u_z^\varepsilon) = t^\varepsilon(z)\omega' + \vartheta_1^\varepsilon(z)e_1 + \vartheta_2^\varepsilon(z)\omega \in T_\omega \mathcal{S} \quad (4.9)$$

$$\oint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot \omega' dx = \oint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot e_1 dx = \oint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot \omega dx = 0, \quad (4.10)$$

and (4.10) shows that *ii*) is fulfilled.

Since integration by parts gives

$$\oint_{\mathbb{S}^1} \omega_z \cdot \omega' dx = 0, \quad \oint_{\mathbb{S}^1} \omega_z \cdot e_1 dx = z_1, \quad \oint_{\mathbb{S}^1} \omega_z \cdot \omega dx = z_2 \oint_{\mathbb{S}^1} |\omega|^2 dx,$$

we can rewrite the orthogonality conditions (4.10) in the following, equivalent way:

$$\oint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega' dx = 0, \quad \oint_{\mathbb{S}^1} u_z^\varepsilon \cdot e_1 dx = z_1, \quad \oint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega dx = z_2 \oint_{\mathbb{S}^1} |\omega|^2 dx. \quad (4.11)$$

Our next aim is to show that $t^\varepsilon(z) = 0$ for any $z \in \overline{\Omega}$, provided that ε is small enough. We have that $\|(u_z^\varepsilon)' - \omega'_z\|_\infty = o(1)$ as $\varepsilon \rightarrow 0$, uniformly for $z \in \overline{\Omega}$. Thus

$$\oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' dx = \oint_{\mathbb{S}^1} \omega'_z \cdot \omega' dx + o(1) = z_2 \oint_{\mathbb{S}^1} |\omega'|^2 dx + o(1).$$

In particular, there exists $\bar{\varepsilon} \in (0, \varepsilon')$ such that $\oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' dx$ is bounded away from 0 if $(\varepsilon, z) \in [-\bar{\varepsilon}, \bar{\varepsilon}] \times \overline{\Omega}$. On the other hand, using (4.3), (4.9), integration by parts and (4.11),

we have

$$\begin{aligned}
0 &= \oint_{\mathbb{S}^1} J_\varepsilon(u_z^\varepsilon) \cdot (u_z^\varepsilon)' dx \\
&= t^\varepsilon(z) \oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' dx + \vartheta_1^\varepsilon(z) \oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot e_1 dx + \vartheta_2^\varepsilon(z) \oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega dx \\
&= t^\varepsilon(z) \oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' dx - \vartheta_2^\varepsilon(z) \oint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega' dx = t^\varepsilon(z) \oint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' dx.
\end{aligned}$$

We see that $t^\varepsilon(z) = 0$ for any $(\varepsilon, z) \in [-\bar{\varepsilon}, \bar{\varepsilon}] \times \bar{\Omega}$, and therefore

$$J_\varepsilon(u_z^\varepsilon) = \vartheta_1^\varepsilon(z) e_1 + \vartheta_2^\varepsilon(z) \omega. \quad (4.12)$$

Now we compute the derivatives of the function $z \mapsto E_{k+\varepsilon K}(u_z^\varepsilon)$ via (4.2) and (4.12). For $j = 1, 2$ we obtain

$$\begin{aligned}
L(u_z^\varepsilon) \partial_{z_j} (E_{k+\varepsilon K}(u_z^\varepsilon)) &= L(u_z^\varepsilon) E'_{k+\varepsilon K}(u_z^\varepsilon) \partial_{z_j} u_z^\varepsilon = \oint_{\mathbb{S}^1} J_\varepsilon(u_z^\varepsilon) \cdot \partial_{z_j} u_z^\varepsilon dx \\
&= \vartheta_1^\varepsilon(z) \oint_{\mathbb{S}^1} \partial_{z_j} u_z^\varepsilon \cdot e_1 dx + \vartheta_2^\varepsilon(z) \oint_{\mathbb{S}^1} \partial_{z_j} u_z^\varepsilon \cdot \omega dx \\
&= \vartheta_1^\varepsilon(z) \partial_{z_j} \left(\oint_{\mathbb{S}^1} u_z^\varepsilon \cdot e_1 dx \right) + \vartheta_2^\varepsilon(z) \partial_{z_j} \left(\oint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega dx \right).
\end{aligned}$$

Then we use (4.11) to infer

$$L(u_z^\varepsilon) \partial_{z_1} (E_{k+\varepsilon K}(u_z^\varepsilon)) = \vartheta_1^\varepsilon(z), \quad L(u_z^\varepsilon) \partial_{z_2} (E_{k+\varepsilon K}(u_z^\varepsilon)) = \vartheta_2^\varepsilon(z) \left(\oint_{\mathbb{S}^1} |\omega|^2 dx \right),$$

that compared with (4.12) give (4.4).

It remains to prove *iv*). Take $z \in \bar{\Omega}$ and consider the function

$$f_z(\varepsilon) = E_{k+\varepsilon K}(u_z^\varepsilon) = E_k(u_z^\varepsilon) + \varepsilon A_K(u_z^\varepsilon), \quad f_z \in C^1(-\bar{\varepsilon}, \bar{\varepsilon}).$$

Clearly $f_z(0) = E_k(\omega_z)$. To compute $f'_z(0)$ notice that $\partial_\varepsilon u_z^\varepsilon$ remains bounded in $C^2(\bar{\Omega})$ as $\varepsilon \rightarrow 0$, because the function $(\varepsilon, z) \mapsto u_z^\varepsilon$ is of class C^1 . Thus $A'_K(u_z^\varepsilon)(\partial_\varepsilon u_z^\varepsilon)$ remains bounded as well. Further, $E'_k(u_z^\varepsilon) \rightarrow E'_k(\omega_z) = 0$ in the norm operator because $u_z^\varepsilon \rightarrow \omega_z$ in C^2 and since ω_z is a k -loop. We infer that

$$f'_z(0) = E'_k(\omega_z)(\partial_\varepsilon u_z^\varepsilon) + A_K(u_z^\varepsilon) + o(1) = A_K(\omega_z) + o(1)$$

uniformly on $\bar{\Omega}$. In fact we proved that

$$f_z(\varepsilon) = E_{k+\varepsilon K}(u_z^\varepsilon) = E_k(\omega_z) + \varepsilon A_K(u_z^\varepsilon) + o(1)$$

uniformly on $\overline{\Omega}$ as $\varepsilon \rightarrow 0$. That is, (4.5) holds true "at the zero order".

To conclude the proof we have to handle $\partial_{z_j}(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z))$ for $j = 1, 2$. Since $J_\varepsilon(u) = J_0(u) + \varepsilon L(u)u_2^{-2}K(u)iu'$, we can rewrite (4.4) as follows,

$$\begin{aligned} \partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon))e_1 + \left(\int_{\mathbb{S}^1} |\omega|^2 dx\right)^{-1} \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon))\omega \\ = \frac{1}{L(u_z^\varepsilon)} J_0(u_z^\varepsilon) + \varepsilon (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) i(u_z^\varepsilon)'. \end{aligned} \quad (4.13)$$

Recall that $J_0(u_z^\varepsilon)$ is orthogonal to e_1 in L^2 , see the second identity in (3.12). We test (4.13) with e_1 to obtain

$$\partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon)) = \varepsilon \int_{\mathbb{S}^1} (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) e_1 \cdot i(u_z^\varepsilon)' dx = \varepsilon A'_K(u_z^\varepsilon) e_1 \quad (4.14)$$

by (2.5). Since $\partial_{z_1}(E_{k+\varepsilon K}(\omega_z)) = \partial_{z_1}(E_k(\omega) + \varepsilon A_K(\omega_z)) = \varepsilon A'_K(\omega_z) e_1$, we get

$$\partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)) = \varepsilon (A'_K(u_z^\varepsilon) e_1 - A'_K(\omega_z) e_1) = o(\varepsilon)$$

because of the continuity of $A'_K(\cdot)$ and since $u_z^\varepsilon \rightarrow \omega_z$.

To handle the derivative with respect to z_2 we test (4.13) with u_z^ε . Since $J_0(u_z^\varepsilon)$ is orthogonal to u_z^ε in L^2 by (3.12), using also (4.11) we obtain

$$z_1 \partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon)) + z_2 \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon)) = \varepsilon \int_{\mathbb{S}^1} (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) u_z^\varepsilon \cdot i(u_z^\varepsilon)' dx = \varepsilon A'_K(u_z^\varepsilon) u_z^\varepsilon,$$

that compared with (4.14) gives

$$z_2 \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon)) = \varepsilon A'_K(u_z^\varepsilon) (u_z^\varepsilon - z_1 e_1).$$

From $z_2 \partial_{z_2}(E_{k+\varepsilon K}(\omega_z)) = z_2 \partial_{z_2}(E_k(\omega) + \varepsilon A_K(\omega_z)) = z_2 \varepsilon A'_K(\omega_z) \omega = \varepsilon A'_K(\omega_z) (\omega_z - z_1 e_1)$, we conclude that

$$z_2 \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)) = \varepsilon (A'_K(u_z^\varepsilon) (u_z^\varepsilon - z_1 e_1) - A'_K(\omega_z) (\omega_z - z_1 e_1)) = o(\varepsilon).$$

The lemma is completely proved. \square

4.2 Existence results

Proof of Theorem 1.1. We are assuming that there exists $r > 0$ such that any function $G \in C^1(\overline{A})$ satisfying $\|G + F_k^K\|_{C^1(\overline{A})} < r$ has a critical point in A . We recall also formula (3.6), that in particular gives

$$E_{k+\varepsilon K}(\omega_z) = E_k(\omega_z) + \varepsilon A_K(\omega_z) = E_k(\omega) - \frac{\varepsilon}{2\pi} F_k^K(z). \quad (4.15)$$

Take an open set $\Omega \in \mathbb{H}^2$ such that $A \Subset \Omega \Subset \mathbb{H}^2$, and let $(\varepsilon, z) \mapsto u_z^\varepsilon$, $(\varepsilon, z) \in [-\bar{\varepsilon}, \bar{\varepsilon}] \times \bar{\Omega}$ be the function given by Lemma 4.2. For $\varepsilon \neq 0$ consider the function

$$G^\varepsilon(z) = \frac{2\pi}{\varepsilon}(E_{k+\varepsilon K}(u_z^\varepsilon) - E_k(\omega))$$

and use (4.15) together with *iv*) in Lemma 4.2 to get

$$\|G^\varepsilon + F_k^K\|_{C^1(\bar{A})} = \frac{2\pi}{|\varepsilon|} \|E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)\| = o(1)$$

as $\varepsilon \rightarrow 0$. We see that for ε small enough the function G^ε has a critical point $z^\varepsilon \in A$. Since the derivatives of the function $z \mapsto E_{k+\varepsilon K}(u_z^\varepsilon)$ vanish at $z = z^\varepsilon$, then $J_\varepsilon(u_{z^\varepsilon}^\varepsilon) = 0$ by (4.4). That is, $u_{z^\varepsilon}^\varepsilon$ is an embedded $k + \varepsilon K$ loop.

The last conclusion in Theorem 1.1 follows via a simple compactness argument and thanks to Theorem 4.1. \square

In the next result we apply Theorem 1.1 to obtain the existence of $k + \varepsilon K$ -loops that shrink to a stable critical point for the curvature function K , as $k \rightarrow \infty$.

Theorem 4.3 *Let $K \in C^1(\mathbb{H}^2)$. Assume that K has a stable critical point in an open set $A \Subset \mathbb{H}^2$. There exists $k_0 > 1$ such that for any $k > k_0$ and for every ε close enough to 0, there exists an embedded $(k + \varepsilon K)$ -loop.*

Moreover, let $k_h \rightarrow \infty, \varepsilon_h \rightarrow 0$ be given sequences. There exist subsequences $k_{h_j}, \varepsilon_{h_j}$, a point $z_\infty \in \bar{A}$ that is critical for K , and an embedded $(k_{h_j} + \varepsilon_{h_j} K)$ -loop w^j such that w^j converges in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ to the constant curve z_∞ , as $h \rightarrow \infty$.

Proof. Recall that $R_k = (k^2 - 1)^{-1/2}$. In order to simplify notations we put

$$z^k := (z_1, kR_k z_2) = z + (kR_k - 1)z_2 e_2 \quad \text{for } z = (z_1, z_2) \in \mathbb{H}^2.$$

Since $D_{\rho_k}^{\mathbb{H}}(z) = D_{R_k z_2}(z^k)$ we have

$$F_k^K(z) = \int_{D_{R_k z_2}(z^k)} p_2^{-2} K(p) dp = \int_{D_{R_k}(0)} (q_2 + kR_k)^{-2} K(z_2 q + z^k) dq. \quad (4.16)$$

We put $\phi_K(q) = q_2^{-2} K(q)$ and rewrite (4.16) as follows:

$$\frac{1}{\pi R_k^2 z_2^2} F_k^K(z) = \oint_{D_{R_k}(0)} \phi_K(z_2 q + z^k) dq.$$

Trivially $kR_k = k/\sqrt{k^2 - 1} \rightarrow 1$ and $|z^k - z| = (kR_k - 1)z_2 \rightarrow 0$ uniformly on \bar{A} , as $k \rightarrow \infty$. Since $\phi_K \in C^1(\mathbb{H}^2)$, it is easy to show that

$$\frac{1}{\pi R_k^2 z_2^2} F_k^K(z) \rightarrow \phi_K(z) = \frac{1}{z_2^2} K(z)$$

in $C^1(\bar{A})$. It follows that for k large enough, F_k^K has a stable critical point in $A \Subset \mathbb{H}^2$. Theorem 1.1 applies and gives the conclusion of the proof. \square

A Loops in the Euclidean plane

The argument we used to prove Theorem 1.1 applies also in the easier Euclidean case. It is well known that the only embedded loops in \mathbb{R}^2 having prescribed constant curvature $k > 0$ are circles of radius $1/k$. We take as a reference circle the loop

$$\omega(x) = \frac{1}{k} x, \quad x \in \mathbb{S}^1 \subset \mathbb{R}^2,$$

that solves

$$u'' = L(u)k i u', \quad \text{where} \quad L(u) := \left(\oint_{\mathbb{S}^1} |u'|^2 dx \right)^{\frac{1}{2}}$$

(in fact, $L(\omega)k = 1$ and $\omega'' = -\omega = i\omega'$).

Let $K \in C^1(\mathbb{R}^2)$ be given. If a nonconstant function $u \in C^2(\mathbb{S}^1, \mathbb{R}^2)$ solves

$$u'' = L(u)(k + \varepsilon K(u)) i u', \tag{A.1}$$

then $|u'| = L(u)$ is constant, and u parameterizes a loop in \mathbb{R}^2 having Euclidean curvature $k + \varepsilon K$ at each point. Further, problem (A.1) admits a variational structure, see [5], [18]. More precisely, its nonconstant solutions are critical points of the energy functional

$$E_{k+\varepsilon K}(u) = \left(\oint_{\mathbb{S}^1} |u'|^2 dx \right)^{\frac{1}{2}} + \varepsilon \oint_{\mathbb{S}^1} Q(u) \cdot i u', \quad u \in C^2(\mathbb{S}^1, \mathbb{R}^2) \setminus \mathbb{R}^2,$$

where the vectorfield $Q \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $\operatorname{div} Q = K$.

Arguing as for Theorem 4.1 one can prove a necessary conditions for the existence of solutions to (A.1) for $\varepsilon = \varepsilon_h \rightarrow 0$.

Theorem A.1 *Let u_h be a $(k + \varepsilon_h K)$ -loop solving (A.1) for $\varepsilon = \varepsilon_h$, and assume that*

$$L(u_h) \rightarrow L_\infty > 0, \quad u_h \rightarrow U \text{ uniformly, for some } U \in C^0(\mathbb{S}^1, \mathbb{R}^2).$$

Then $U(x) = \omega(\xi x^\mu) + z$ for some $\mu \in \mathbb{N}$, $\xi \in \mathbb{S}^1$ and $z \in \mathbb{R}^2$, that is a critical point for the Melnikov function

$$F_k^K(z) = \int_{D_{\frac{1}{k}}(z)} K(q) dq, \quad F_k^K : \mathbb{R}^2 \rightarrow \mathbb{R}.$$

In the Euclidean case we have the following existence result.

Theorem A.2 *Let $k > 0$ and $K \in C^1(\mathbb{R}^2)$ be given. Assume that F_k^K has a stable critical point in an open set $A \Subset \mathbb{R}^2$. Then for every $\varepsilon \in \mathbb{R}$ close enough to 0, there exists an embedded $(k + \varepsilon K)$ -loop $u^\varepsilon : \mathbb{S}^1 \rightarrow \mathbb{R}^2$.*

Moreover, any sequence $\varepsilon_h \rightarrow 0$ has a subsequence ε_{h_j} such that $u^{\varepsilon_{h_j}} \rightarrow \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ as $j \rightarrow \infty$, where $z_0 \in A$ is a critical point for F_k^K .

Sketch of the proof. We introduce the 3-dimensional space of embedded solutions to the unperturbed problem, namely

$$\mathcal{S} = \{\omega \circ \xi + z \mid \xi \in \mathbb{S}^1, z \in \mathbb{R}^2\},$$

and the functions $J_\varepsilon : C^2(\mathbb{R}, \mathbb{R}^2) \setminus \mathbb{R}^2 \rightarrow C^0(\mathbb{R}, \mathbb{R}^2)$, $\varepsilon \in \mathbb{R}$, given by

$$J_\varepsilon(u) = -u'' + L(u)(k + \varepsilon K(u))iu' = J_0(u) + L(u)K(u)iu'.$$

We have $\mathcal{S} \subset \{J_0 = 0\}$. Since $J'_0(\omega + z)\varphi = -\varphi'' + i\varphi' - k^2 \left(\oint_{\mathbb{S}^1} \varphi \cdot \omega dx \right) \omega$, it is quite easy to check that

$$T_{\omega+z}\mathcal{S} = \langle \omega', e_1, e_2 \rangle = \ker J'_0(\omega + z),$$

and that $J'_0(\omega + z) : T_{\omega+z}\mathcal{S}^\perp \rightarrow T_{\omega+z}\mathcal{S}^\perp$ is invertible. The remaining part of the proof runs with minor changes. \square

Theorem 4.3 has its Euclidean correspondent as well. We omit the proof of the next result.

Theorem A.3 *Let $K \in C^1(\mathbb{R}^2)$. Assume that K has a stable critical point in an open set $A \subseteq \mathbb{R}^2$. Then there exists $k_0 > 1$ such that for any fixed $k > k_0$, and for every ε close enough to 0, there exists an embedded $(k + \varepsilon K)$ -loop $u^{k,\varepsilon} : \mathbb{S}^1 \rightarrow \mathbb{R}^2$.*

Moreover, there exist sequences $k_h \rightarrow \infty$, $\varepsilon_h \rightarrow 0$ such that $u^{k_h, \varepsilon_h j} \rightarrow \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ as $j \rightarrow \infty$, where $z_0 \in A$ is a critical point for K .

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